

## ALMOST CLASSICALLY DAMPED LINEAR DISCRETE SYSTEMS

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### ABSTRACT

The present work investigates dynamic response of a class of linear oscillators. The common characteristic of the systems analyzed is that they possess damping properties close to those resulting in classical normal modes. Regular perturbation expansions are utilized for analyzing the eigenproblem as well as the vibration response of such systems. The analysis is based on a proper splitting of the damping matrix. The advantage of this approach is that it sets the stage for application of standard modal analysis methodologies, reducing the main mathematical problem to that of finding the frequencies and mode shapes of the corresponding undamped model. The validity and effectiveness of the present analysis is illustrated and verified by a numerical example.

### 1. INTRODUCTION

In many instances, the dynamic response of a mechanical system can be accurately predicted by a linear discrete model. When damping effects are neglected, the equations of motion of these systems can be reduced into a set of uncoupled equations representing single degree of freedom oscillators. However, introduction of damping allows this uncoupling only for some special combinations of the damping parameters [1, 2]. In such cases, the system possesses classical normal modes and is called classically damped. In practice, these conditions are not met exactly (e.g., [3]). Then, the system response is usually obtained by either direct integration or by properly converting the system equations to first order form (e.g., [4, 5]).

The present work focuses on the response of systems which do not possess classical normal modes, but the damping properties are close to those meeting the conditions leading to normal modes. These systems are called almost classically damped and present a lot of practical significance. As a result, various aspects of their response have already been examined before by other investigators. For example, Chung and Lee [6] extended

a perturbation methodology developed earlier by Meirovitch and Ryland [7] in order to obtain the eigensolution of discrete systems. Also, Cronin [8] presented a perturbation analysis for the response of such systems under harmonic excitation, while Udwadia and Esfandiari [9] presented an iterative approach for obtaining the response under general forcing. Finally, more references and information about damping effects can be found in the books [10-12].

The objective of the present investigation is to generalize previous work on almost classically damped oscillators. This is done by developing approximate but sufficiently accurate methodologies for determining response characteristics of such systems under arbitrary forcing conditions. Standard regular perturbation techniques are employed, coupled with classical modal analysis formulations. As a result, the solution is obtained by utilizing only the real modes and frequencies of the corresponding undamped system. This is especially efficient in cases where the solution of a classically damped system is known and the effect of small changes in the damping parameters is sought. The analysis is based on a proper decomposition of the damping matrix. The applicability and accuracy of the present analysis is demonstrated by practical examples.

### 3. ANALYSIS

The equations of motion of linear discrete viscously damped

$$M \ddot{\underline{x}} + C \dot{\underline{x}} + K \underline{x} = \underline{f}(t). \quad (1)$$

Here  $M$ ,  $C$  and  $K$  are  $N \times N$  real symmetric and positive definite matrices, while  $\underline{x}(t)$  and  $\underline{f}(t)$  represent the generalized displacement and forcing vectors, respectively. The analysis starts by first considering the undamped problem. Namely, let  $\lambda_n$  be an eigenvalue and  $\underline{x}_n$  be the corresponding eigenvector, i.e.

$$(\lambda_n^2 M + K) \underline{x}_n = \underline{0}.$$

For the class of problems considered, the  $\lambda_n$ 's are purely imaginary and the  $\underline{x}_n$ 's can be chosen to be real. Then, assume that the damping matrix  $C$  can be decomposed according to:

$$C = C_0 + \varepsilon C_1, \quad (2)$$

with  $|\varepsilon| \ll 1$  and  $C_0$  chosen so that it leads to classical normal modes and the diagonal elements of  $C_1$  are zero. Next, construct the modal matrix  $X = [\underline{x}_1 \cdots \underline{x}_N]^T$  so that

$$X^T M X = I, \quad X^T K X = \Omega, \quad X^T C_0 X = Z, \quad (3)$$

where  $I$  is the  $N \times N$  identity matrix, while  $\Omega$  and  $Z$  are diagonal matrices with elements  $\omega_n^2 = -\lambda_n^2$  and  $2\zeta_n \omega_n$  ( $n=1, \dots, N$ ), respectively. Employing (3) and using the identity  $\underline{x}_n = X \underline{e}_n$ , where  $\underline{e}_n$  is the  $n^{\text{th}}$  unit vector in an  $N$ -dimensional Euclidean space, it can easily be shown that

$$(\rho_n^2 M + \rho_n C_0 + K) \underline{x}_n = \underline{0}, \quad (4)$$

provided that  $\rho_n$  is chosen from

$$\rho_n^2 + 2\omega_n \zeta_n \rho_n + \omega_n^2 = 0.$$

These imply that  $\underline{x}_n$  remains an eigenvector of the classically damped system, while the corresponding eigenvalues  $\lambda_n = \pm i \omega_n$  become complex, with form

$$\rho_n = -\omega_n \zeta_n \pm i \omega_n \sqrt{1 - \zeta_n^2}.$$

In case of nonclassically damped systems, all or some of the off-diagonal elements of the matrix

$$\hat{C} = X^T C_1 X$$

are nonzero. In cases of small  $\varepsilon$  approximate procedures can be applied, as shown next.

### Approximate Solution of the Eigenproblem

When  $\varepsilon$  is small in (2), the changes of the eigensolution can be accounted for with approximate approaches (see [6, 7]). Here, an alternative method is developed, which is somewhat simpler and attacks the problem in the second rather than the first order form. Namely, for a damping matrix of the form expressed by (2), the eigenvalues and eigenvectors of the original system, which satisfy

$$(s_n^2 M + s_n C + K) \underline{x}_n = \underline{0}, \quad (5)$$

are expressed to second order in  $\varepsilon$  as follows:

$$s_n = \rho_n + \varepsilon \sigma_n + \varepsilon^2 \tau_n + O(\varepsilon^3), \quad (6)$$

$$\underline{x}_n = \underline{x}_n + \varepsilon \underline{y}_n + \varepsilon^2 \underline{z}_n + O(\varepsilon^3). \quad (7)$$

To determine the perturbations, substitute (2), (6) and (7) in (5) and collect terms with the same order of  $\varepsilon$ . This yields

$$(\rho_n^2 M + \rho_n C_0 + K) \underline{x}_n = \underline{0}, \quad (8)$$

$$(\rho_n^2 M + \rho_n C_0 + K) \underline{y}_n = -(2\rho_n \sigma_n M + \rho_n C_1 + \sigma_n C_0) \underline{x}_n, \quad (9)$$

$$(\rho_n^2 M + \rho_n C_0 + K) \underline{z}_n = -(2\rho_n \sigma_n M + \rho_n C_1 + \sigma_n C_0) \underline{y}_n - [(2\rho_n \tau_n + \sigma_n^2) M + \sigma_n C_1 + \tau_n C_0] \underline{x}_n. \quad (10)$$

The first of these equations is identical to (4), as expected. Then, let

$$\underline{y}_n = X \underline{u}_n, \quad (11)$$

premultiply (9) by  $X^T$  and use (3) to obtain a set of  $N$  uncoupled algebraic equations for the components of  $\underline{u}_n$  of the form:

$$\Pi_n \underline{u}_n = -Q_n \underline{e}_n, \quad (12)$$

where the matrices  $\Pi_n = [\pi_{npm}]$  and  $Q_n = [q_{npm}]$  are defined by

$$\Pi_n = \rho_n^2 I + \rho_n Z + \Omega \quad \text{and} \quad Q_n = 2\rho_n \sigma_n I + \sigma_n Z + \rho_n \hat{C}.$$

$\Pi_n$  is diagonal and by the definition of  $\rho_n$  its  $n^{\text{th}}$  diagonal element is zero. Then, choose

$$\sigma_n = 0 \quad (13)$$

$u_{nn} = 0$  and the remaining components of  $u_{nn}$  from (12), by simply dividing  $\rho_n \hat{c}_{mn}$  by  $\pi_{nmm}$ . Likewise, let:

$$\underline{z}_n = X \underline{w}_n. \quad (14)$$

premultiply (10) by  $X^T$  and obtain a new set of uncoupled algebraic equations for  $\underline{w}_n$ :

$$\Pi_n \underline{w}_n = -Q_n \underline{u}_n - (2\rho_n I + Z) \tau_n \underline{e}_n. \quad (15)$$

Again, choose  $w_{nn} = 0$ , the second order correction of the eigenvalue from

$$\tau_n = - \left[ \sum_{m=1}^n q_{nmm} u_{nm} \right] / [2(\rho_n + \zeta_n \omega_n)] \quad (16)$$

and the remaining components of  $\underline{w}_n$  from (15).

### Approximate Solution of the Response Problem

Following standard modal analysis procedures, the general solution of the response problem can also be treated in a similar way. Namely, the solution of (1) is first expressed in the form

$$\underline{x}(t) = X \underline{\eta}(t). \quad (17)$$

Then, premultiply (1) by  $X^T$  and use (2), (3) and (17) to get

$$\ddot{\underline{\eta}} + (Z + \varepsilon \hat{C}) \dot{\underline{\eta}} + \Omega \underline{\eta} = \underline{h}(t).$$

Next, let

$$\underline{\eta}(t) = \sum_{n=0}^k \varepsilon^n \underline{\eta}_n(t) + O(\varepsilon^{k+1}). \quad (18)$$

Substituting  $\underline{\eta}$  from (18) in the previous equation and collecting terms of the same order in  $\varepsilon$  results in

$$\ddot{\underline{\eta}}_n + Z \dot{\underline{\eta}}_n + \Omega \underline{\eta}_n = \underline{h}_n(t). \quad (19)$$

with

$$\underline{h}_0(t) = X^T \underline{f}(t) \quad \text{and} \quad \underline{h}_n(t) = -\hat{C} \dot{\underline{\eta}}_{n-1}(t), \quad \text{for } n=1, \dots, k.$$

Obviously, (19) represents a set of uncoupled equations of single degree of freedom oscillators. Imposing a set of initial conditions will give

$$\underline{\eta}(0) = X^T M \underline{x}(0) \quad \text{and} \quad \dot{\underline{\eta}}(0) = X^T M \dot{\underline{x}}(0),$$

which completes the solution of (19) for  $\underline{u}_n$  and the subsequent determination of the response  $\underline{x}(t)$  from (17) and (18), under general forcing conditions.

### 3. EXAMPLE

The example system considered is shown in Fig. 1. The displacement  $x_1$  is absolute, while  $x_2$  represents the displacement of  $m_2$  relative to  $m_1$ . Introducing the dimensionless param-

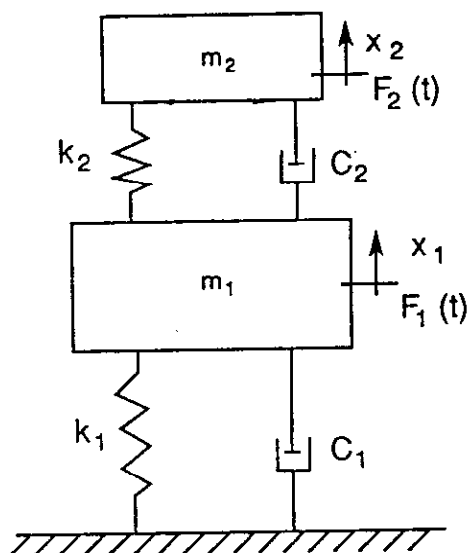


Fig. 1: Model of the Example System.

eters:  $\bar{\omega}_i = \sqrt{k_i/m_i}$ ,  $\zeta_i = c_i/(2\sqrt{k_i m_i})$ , ( $i=1,2$ ),  $\tau = \bar{\omega}_1 t$ ,  $\rho = \bar{\omega}_2/\bar{\omega}_1$  and  $\mu = m_2/m_1$ , the mass, damping and stiffness matrices are expressed by

$$M = \begin{bmatrix} 1+\mu & \mu \\ \mu & \mu \end{bmatrix}, \quad C = \begin{bmatrix} 2\zeta_1 & 0 \\ 0 & 2\mu\rho\zeta_2 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & \mu\rho^2 \end{bmatrix},$$

while the forcing vector is given by

$$\underline{f} = \begin{bmatrix} F_1 + F_2 \\ F_2 \end{bmatrix} / k_1 x_c.$$

Choosing  $C_o = \alpha K$  preserves normal modes. If  $\zeta_{Ni}$  are damping ratios leading to normal modes, the above requires that  $\zeta_{N2} = \rho \zeta_{N1}$ . Here, choose:

$$\zeta_1 = \zeta_{N1} \quad \text{and} \quad \zeta_2 = (1 + \epsilon) \rho \zeta_{N1}.$$

First, consider the eigenvalue problem for a system with parameters  $\zeta_1 = 0.1 = \mu$  and  $\rho = 1$ . Fig. 2(a) shows the real part of the lower eigenvalue  $s_1$  of the system, while Fig. 2(b) shows the imaginary part of the same eigenvalue, for

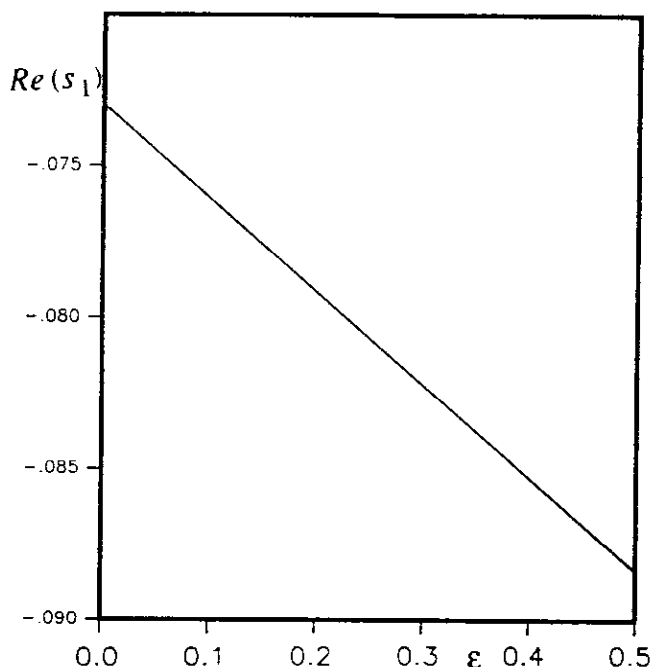


Fig. 2(a): Dependence of Real Part of  $s_1$  on  $\epsilon$ .

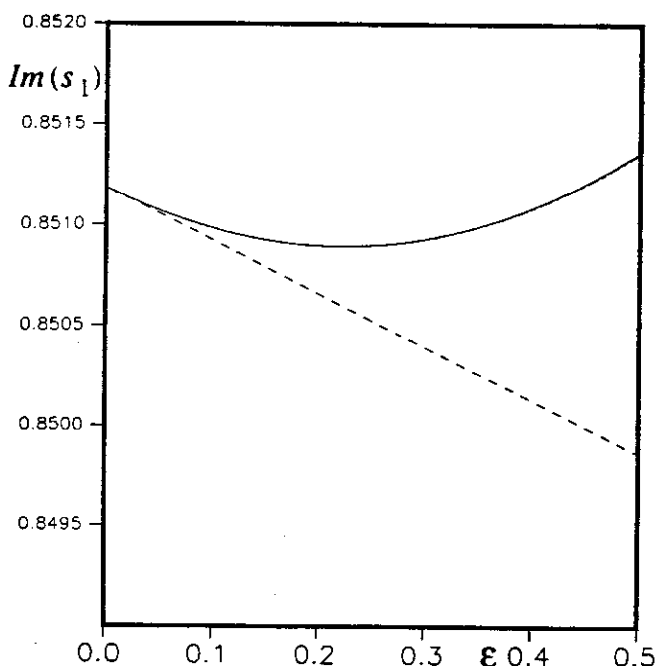


Fig. 2(b): Dependence of Imaginary Part of  $s_1$  on  $\epsilon$ .

$0 \leq \epsilon \leq 0.5$ . Solid lines represent exact values, while broken/dotted lines correspond to values obtained from first/second order perturbation solutions. Obviously, the approximate and exact values are almost identical for the real

part, while second order perturbation gives almost exact results for the imaginary part also. Similar results were obtained for the second eigenvalue.

The eigenvectors are normalized so that their first component is real and equal to the first component of the corresponding M-normalized eigenvector of the undamped problem. Then, their second component is a complex constant. Figures 3(a) and 3(b) use the same line representation as that of Fig. 2(a) to show the dependence of the real and imaginary part of the second component for the eigenvector  $\underline{\hat{x}}_1$ , corresponding to the lower frequency  $s_1$ . Similar results were also obtained for the other eigenvector.

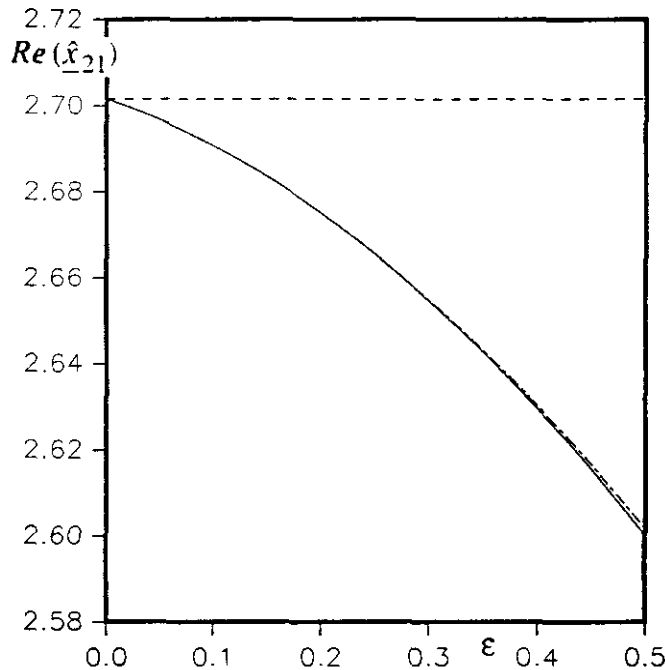


Fig. 3(a): Dependence of Real Part of  $\underline{\hat{x}}_{21}$  on  $\epsilon$ .

Next, consider forced response with  $\epsilon = 0.5$  and constant forcing. First, harmonic excitation is considered with  $F_1(t) = F_1 \cos \Omega t$ ,  $F_2 = 0$  and  $\zeta_1 = 0.05$ . Introducing the characteristic length  $x_c = F_1/k_1$  and the normalized frequency  $\omega = \Omega/\omega_1$ , the components of the forcing vector become:  $f_1 = \cos \omega t$  and  $f_2 = 0$ . Figure 4(a) shows the response diagram of the mass  $m_1$ , while Fig. 4(b) shows the phase angle  $\phi_1$  of  $m_1$  with respect to the forcing. Here, the broken line shows results obtained by neglecting the off diagonal terms of the damping matrix (i.e. for  $\epsilon = 0$ ). Comparison with the exact values (continuous line) shows inaccuracies around both the resonance frequencies as well as the intermediate range. The dotted lines show results including the first order corrections.

Finally, transient excitation with constant  $F_1$  and  $F_2$  is con-

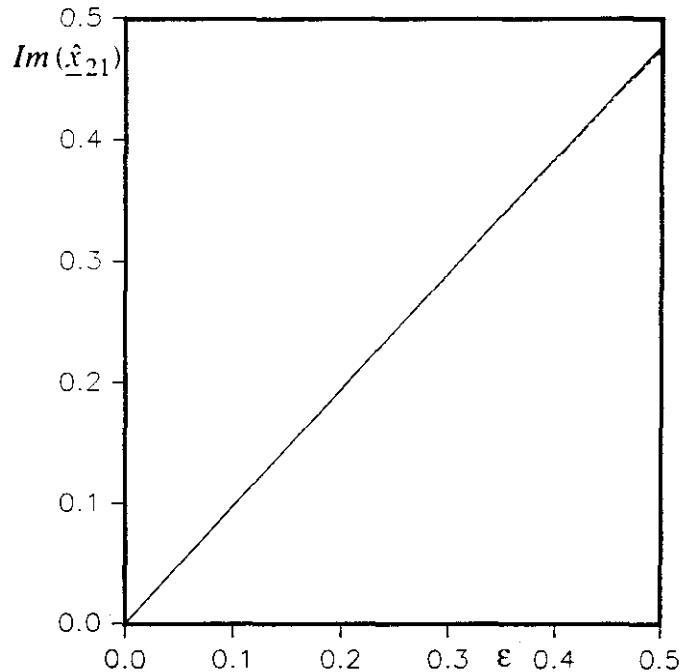


Fig. 3(b): Dependence of Imaginary Part of  $\underline{\hat{x}}_{21}$  on  $\epsilon$ .

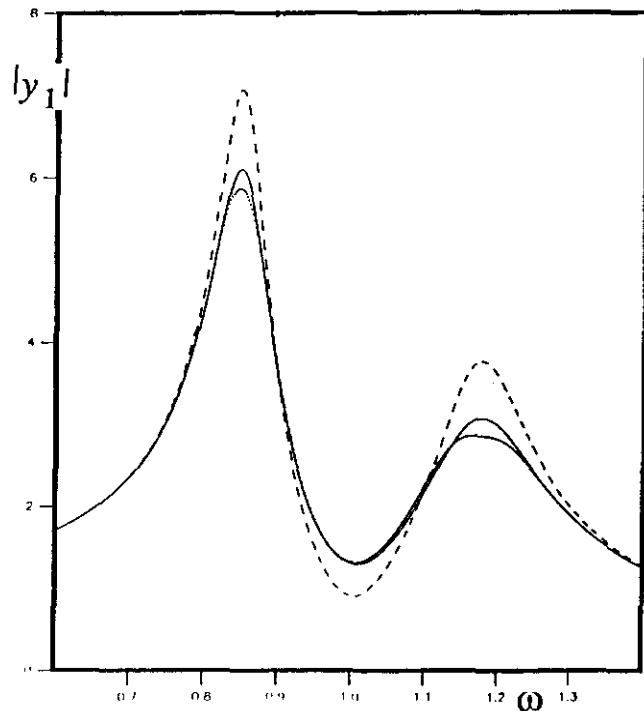


Fig. 4(a): Response Diagram for  $m_1$  ( $\epsilon = 0.5$ ).

sidered. Here,  $f_1 = 1+p$  and  $f_2 = p$ , with  $p = F_2/F_1$ . Figures 5(a) and 5(b) show the response history of  $m_1$  and  $m_2$ , respectively, for  $p = 1$ ,  $\zeta_1 = 0.1$  and starting from zero initial condi-

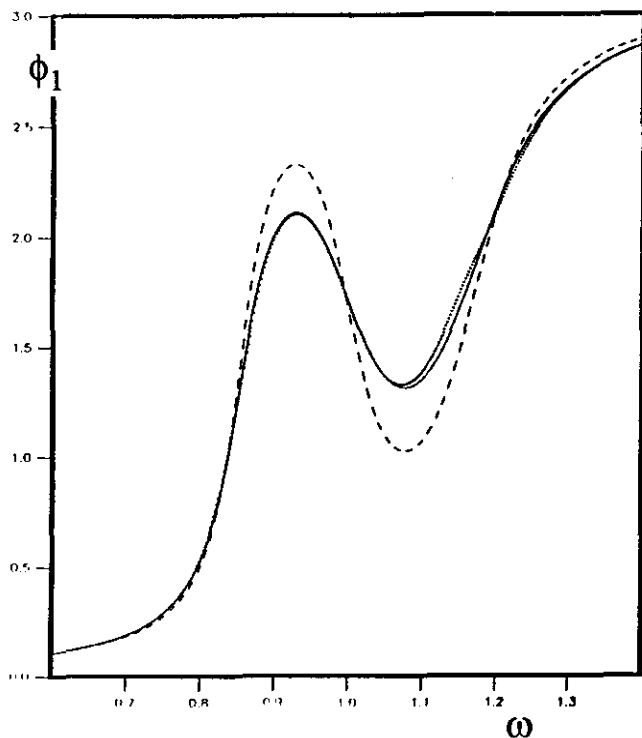


Fig. 4(b): Phase Diagram for  $m_1$  ( $\epsilon = 0.5$ ).

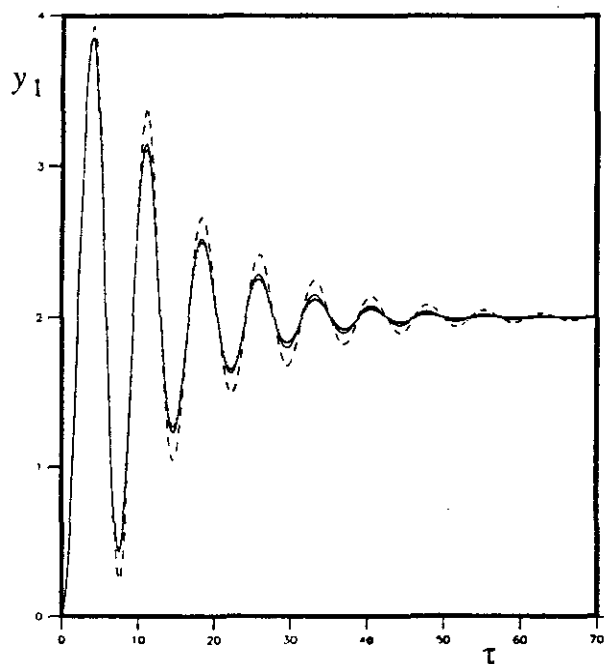


Fig. 5(a): Response History of  $m_1$  ( $\epsilon = 0.5$ ).

tions. Broken/dotted lines correspond to results obtained from the first/second order perturbation solution, while the continuous lines represent the exact solution.

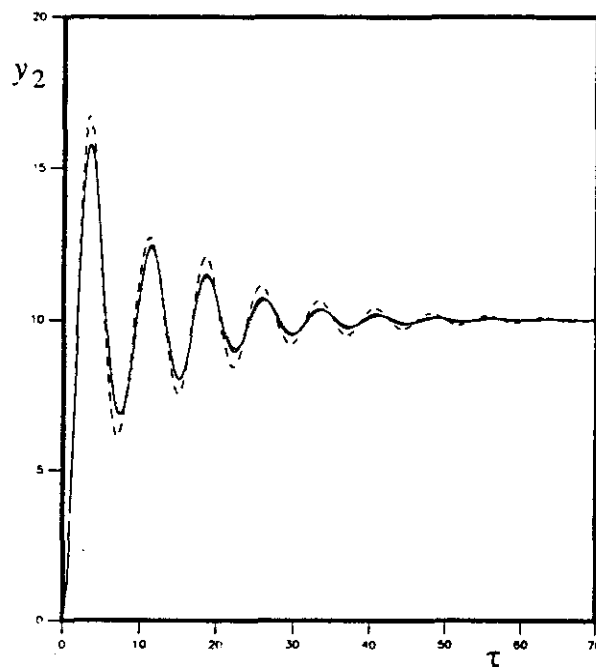


Fig. 5(b): Response History of  $m_2$  ( $\epsilon = 0.5$ ).

#### 4. SUMMARY

A perturbation analysis is developed for the response of almost classically damped linear oscillators. The applied methodology provides approximate but sufficiently accurate analytical solutions for the eigenproblem as well as the response under general forcing conditions. The idea is based on a decomposition of the damping matrix into a part that results in normal mode response - with arbitrarily large damping ratios - plus a small perturbation. This permits an effective application of standard modal analysis methodologies and provides the solution of the problem in terms of the frequencies and modes of the corresponding undamped problem only. In turn, this is expected to result in computational benefits, especially when performing parametric studies to investigate damping effects for design purposes. Finally, the validity and accuracy of the approach is confirmed by numerical examples. A similar methodology is currently extended for continuous systems, where the effect of both the field equation and the boundary conditions need to be taken into account in performing the decomposition of the damping effects.

#### 5. REFERENCES

- [1] Caughey, T.K., 1960, "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 27, pp. 269-271.
- [2] Caughey, T.K. and O'Kelly, M.E.J., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 32, pp. 583-

588.

[3] Natsiavas, S., 1993, "Dynamics of Multiple Degree of Freedom Oscillators with Colliding Components," *Journal of Sound and Vibration*, (in press).

[4] Foss, K.A., 1958, "Coordinates which Uncouple the Equations of Motion of Damped Linear Dynamic Systems," *ASME Journal of Applied Mechanics*, Vol. 25, pp. 361-364.

[5] Meirovitch, L., 1967, *Analytical Methods in Vibrations*, Macmillan, New York.

[6] Chung, K.R. and Lee, C.W., 1986, "Dynamic Reanalysis of Weakly Nonproportional Damped Systems," *Journal of Sound and Vibration*, Vol. 111, pp. 37-50.

[7] Meirovitch, L. and Ryland, G., II, 1979, "Response of Slightly Damped Gyroscopic Systems," *Journal of Sound and Vibration*, Vol. 67, pp. 1-19.

[8] Cronin, D.L., 1976, "Approximation for Determining Harmonically Excited Response of Nonclassically Damped Systems," *ASME Journal of Engineering for Industry*, Vol. 98, pp. 43-47.

[9] Udawadia, F.E. and Esfandiari, R.S., 1990, "Nonclassically Damped Dynamic Systems: An Iterative Approach," *ASME Journal of Applied Mechanics*, Vol. 57, pp. 423-433.

[10] Snowdon, J.C., 1968, *Vibration and Shock in Damped Mechanical Systems*, J. Wiley and Sons, New York.

[11] Nashif, A.D., Jones, D.I.G. and Henderson, J.P., 1985, *Vibration Damping*, J. Wiley and Sons, New York.

[12] Inman, D.J., 1989, *Vibration with Control, Measurement and Stability*, Prentice Hall, Englewood Cliffs, New Jersey.